RMSC 4003
Statistical Modeling in Financial Markets
Tutorial 10 Solution

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1 Examples on Ito's Lemma

Example 1.1. The geometric mean reversion process X_t is defined as the solution of the stochastic differential equation

$$dX_t = K(\alpha - \log X_t)X_t dt + \sigma X_t dW_t; \quad X_0 = x > 0,$$

where K, α, σ and x are positive constants.

(a) Find dY_t, where $Y_t = \log X_t$.

(b) Solve Y_t.

(c) Show that $X_t = \exp\left\{e^{-Kt}\log x + (\alpha - \frac{\sigma^2}{2K})(1-e^{-Kt}) + \sigma e^{-Kt}\int_0^t e^{Ks}dW_s\right\}$.

Solution. (a) Let $f(x,t) = \log x$. Then $f_t = 0, f_x = \frac{1}{x}$ and $f_{xx} = -\frac{1}{x^2}$. Note that $a = K(\alpha - \log X_t)X_t$ and $b = \sigma X_t$. By Ito's lemma,

$$\begin{aligned} d\log X_t &= \left(K(\alpha - \log X_t)X_t\frac{1}{X_t} + \frac{1}{2}(\sigma X_t)^2(-\frac{1}{X_t^2})\right)dt + \sigma X_t\frac{1}{X_t}dW_t \\ &= K(\alpha - \frac{\sigma^2}{2K} - \log X_t)dt + \sigma dW_t. \end{aligned}$$

That is

$$dY_t = K(\alpha - \frac{\sigma^2}{2K} - Y_t)dt + \sigma dW_t.$$

(b) Let $f(x,t) = xe^{Kt}$. Then $f_t = xKe^{Kt}, f_x = e^{Kt}$ and $f_{xx} = 0$. Note that $a = K(\alpha - \frac{\sigma^2}{2K} - Y_t)$ and $b = \sigma$. By Ito's lemma,

$$\begin{aligned} d(Y_te^{Kt}) &= \left(KY_te^{Kt} + K(\alpha - \frac{\sigma^2}{2K} - Y_t)e^{Kt}\right)dt + \sigma e^{Kt}dW_t \\ &= K(\alpha - \frac{\sigma^2}{2K})e^{Kt}dt + e^{Kt}\sigma dW_t \\ Y_te^{Kt} - Y_0 &= \int_0^t K(\alpha - \frac{\sigma^2}{2K})e^{Ks}ds + \int_0^t e^{Ks}\sigma dW_s \\ Y_t &= Y_0e^{-Kt} + K(\alpha - \frac{\sigma^2}{2K})e^{-Kt}\int_0^t e^{Ks}ds + e^{-Kt}\int_0^t e^{Ks}\sigma dW_s \\ &= e^{-Kt}\log x + e^{-Kt}(\alpha - \frac{\sigma^2}{2K})(e^{Kt}-1) + \sigma e^{-Kt}\int_0^t e^{Ks}dW_s \\ &= e^{-Kt}\log x + (\alpha - \frac{\sigma^2}{2K})(1-e^{-Kt}) + \sigma e^{-Kt}\int_0^t e^{Ks}dW_s. \end{aligned}$$

(c) Exponentiating, we get

$$X_t = \exp\left\{e^{-Kt}\log x + (\alpha - \frac{\sigma^2}{2K})(1-e^{-Kt}) + \sigma e^{-Kt}\int_0^t e^{Ks}dW_s\right\}.$$

Example 1.2. Evaluate

$$\int_0^T e^{-t}W_t dW_t.$$

Solution. Let $f(t,x) = e^{-t}x^2$. Then $f_t = -e^{-t}x^2, f_x = 2e^{-t}x$ and $f_{xx} = 2e^{-t}$. Note that $a = 0$ and $b = 1$. By Ito's lemma

$$d(e^{-t}W_t^2) \quad = \quad (-e^{-t}W_t + e^{-t})dt + 2e^{-t}W_t dW_t.$$

Integrating both sides,

$$e^{-T}W_T^2 = -\int_0^T e^{-t}W_t dt + (1-e^{-T}) + 2\int_0^T e^{-t}W_t dW_t.$$

Rearranging the terms,

$$\int_0^T e^{-t}W_t dW_t = \frac{1}{2}e^{-T}W_T^2 - \frac{1}{2}(1-e^{-T}) + \frac{1}{2}\int_0^T e^{-t}W_t dt.$$

Example 1.3. (2011-2012 Final Q3)

(a) Suppose that the stock price is governed by the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where W_t is the Wiener process. Fine the stochastic differential equation that governs $e^{-\mu t}S_t$.

(b) Consider the stochastic differential equation

$$dX_t = \mu(10 - X_t)dt + \sigma dW_t.$$

Show that the solution to the stochastic differential equation has the form

$$X_t = g(t) + 10\mu\int_0^t e^{-\mu(t-u)}du + \sigma\int_0^t e^{-\mu(t-u)}dW_u.$$

Here $g(t)$ is some non-random function. Further assume that $X_0 = 12$. Find $g(t)$.

Solution. (a) Let $f(x,t) = e^{-\mu t}x$. Then $f_t = -\mu e^{-\mu t}x,\ f_x = e^{-\mu t}$ and $f_{xx} = 0$. Note that $a = \mu S_t$ and $b = \sigma S_t$. By Ito's lemma,

$$\begin{aligned} d(e^{-\mu t}S_t) &= (-\mu e^{-\mu t}S_t + e^{-\mu t}\mu S_t)dt + e^{-\mu t}\sigma S_t dW_t) \\ &= e^{-\mu t}\sigma S_t dW_t. \end{aligned}$$

(b) Let $f(x,t) = e^{\mu t}x$. Then $f_t = \mu e^{\mu t}x,\ f_x = e^{\mu t}$ and $f_{xx} = 0$. Note that $a = \mu(10 - X_t)$ and $b = \sigma$. By Ito's lemma,

$$\begin{aligned} d(e^{\mu t}X_t) &= (\mu e^{\mu t}X_t + e^{\mu t}\mu(10 - X_t))dt + e^{\mu t}\sigma dW_t \\ &= 10e^{\mu t}\mu dt + e^{\mu t}\sigma dW_t \\ e^{\mu t}X_t - X_0 &= \int_0^t 10e^{\mu u}\mu du + \int_0^t e^{\mu u}\sigma dW_u \\ X_t &= X_0 e^{-\mu t} + \int_0^t 10e^{-\mu(t-u)}du + \sigma\int_0^t e^{-\mu(t-u)}dW_u. \end{aligned}$$

So $g(t) = 12e^{-\mu t}$.

2 Brief Introduction to Simulation of Stock Price Process

If the stock price process S_t satisfies

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

we know the exact solution is

$$S_t = S_0 \exp\{(\mu - \frac{\sigma^2}{2})t + \sigma W_t\}.$$

We can carry out the simulation as follows:

$$\begin{aligned} S_{t+\Delta t} - S_t &= \mu S_t \Delta t + \sigma S_t (W_{t+\Delta t} - W_t) \\ &= \mu S_t \Delta t + \sigma S_t \sqrt{\Delta t} Z_t, \end{aligned}$$

where $Z_t \sim N(0,1)$, as $W_{t+\Delta t} - W_t \sim N(0, \Delta t)$. Thus, we can proceed our simulation using

$$\begin{aligned} S_{t+\Delta t} &= S_t + \mu S_t \Delta t + \sigma S_t \sqrt{\Delta t} Z_t \\ &= (1 + \mu \Delta t + \sigma \sqrt{\Delta t} Z_t) S_t. \end{aligned}$$

Similarly, if we start with $d \log S_t = (\mu - \frac{\sigma^2}{2})dt + \sigma dW_t$, we have the following:

$$\log S_{t+\Delta t} = \log S_t + (\mu - \frac{\sigma^2}{2})\Delta t + \sigma \sqrt{\Delta t} Z_t.$$

That is,

$$S_{t+\Delta t} = \exp\{(\mu - \frac{\sigma^2}{2})\Delta t + \sigma \sqrt{\Delta t} Z_t\} S_t.$$

3 Black-Scholes formula

This section will not be tested. Suppose the stock price process follows the geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dW_t.$$

Consider a European call option with strike price K and expiry T. Assume that the underlying stock pays no dividends during the time $[0, T]$ and the continuously compounding risk-free rate is r. The price c of the call option at time 0 is

$$c = S_0 \Phi(d_1) - K e^{-rT} \Phi(d_2),$$

where

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T}} \left[\log(\frac{S_0}{K}) + (r + \frac{\sigma^2}{2})T \right] \\ d_2 &= d_1 - \sigma\sqrt{T} = \frac{1}{\sigma\sqrt{T}} \left[\log(\frac{S_0}{K}) + (r - \frac{\sigma^2}{2})T \right]. \end{aligned}$$

Example 3.1. Derive the Black-Scholes Formula using risk-neutral valuation.

Solution. From risk-neutral valuation, we know that

$$c = e^{-rT}\, \mathrm{E}(\max(S_T - K, 0)),$$

where

$$dS_t = rS_t dt + \sigma S_t dW_t.$$

Therefore, $S_T = S_0 \exp\{(r - \frac{1}{2}\sigma^2)T + \sigma W_T\} = S_0 \exp\{(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}Z\}$, where $Z \sim N(0,1)$. Note that

$$\begin{aligned} & S_0 \exp\{(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}x\} > K \\ \iff \quad & x > \frac{\log(\frac{K}{S_0}) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = \frac{-\log(\frac{S_0}{K}) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = -d_2. \end{aligned}$$

Thus,

$$\begin{aligned} c &= e^{-rT}\int_{-\infty}^{\infty} (S_0 \exp\{(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}x\} - K)^+ \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \\ &= e^{-rT}\int_{-d_2}^{\infty} S_0 \exp\{(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}x\} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx - e^{-rT}K \int_{-d_2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \\ &= S_0 \int_{-d_2}^{\infty} e^{-\frac{1}{2}\sigma^2 T + \sigma\sqrt{T}x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx - Ke^{-rT}(1 - \Phi(-d_2)) \\ &= S_0 \int_{-d_2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x - \sigma\sqrt{T})^2}{2}} dx - Ke^{-rT}\Phi(d_2) \\ &= S_0 \int_{-d_2 - \sigma\sqrt{T}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy - Ke^{-rT}\Phi(d_2) \\ &= S_0(1 - \Phi(-d_1)) - Ke^{-rT}\Phi(d_2) \\ &= S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2). \end{aligned}$$

A sample R function to compute the option price under Black-Scholes model:

```
BSprice<- function(S0,K,r,sigma,T,type) {
# Call option: type = 1
# Put option: type = -1
d1 <- (log(S0/K) + (r + 0.5*sigma^2)*T)/sigma/sqrt(T)
d2 <- (log(S0/K) + (r - 0.5*sigma^2)*T)/sigma/sqrt(T)
price <- type*S0*pnorm(type*d1,0,1) - type*K*exp(-r*T)*pnorm(type*d2,0,1)
return(price)
}
```

Let $S_0 = 100, r = 0.05, \sigma = 0.3$. Suppose you want to determine the price of a European put option maturing in 20 days with a strike price $K = 105$.

```
BSprice(100,105,0.05,0.3,20/250,-1)
```

Thank you for attending my tutorial.